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Application of Conjugate Gradient Method with Cubic Non-Polynomial Spline Scheme for Solving Two-Point Boundary Value Problems

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ABSTRACT

Objective – Conjugate Gradient (CG) method is used to solve two-point boundary value problems together with non-polynomial spline approach at cubic degree.
Methodology/Technique – To develop a system of linear equations in a matrix form, cubic non-polynomial splines are used to descretize the two-point boundary value problems so that the approximation can be computed using CG method. Since many previous researchers attempt to obtain the approximate solution for the two-point boundary value-problems at different degree of non-polynomial splines only, then the present paper aims to look into method which is best used with the cubic non-polynomial splines in order to approximate the solution of these problem
Findings – According to the performance analysis results in term of iterations number, execution time and maximum absolute error at different grid sizes, the application of CG method together with the cubic non-polynomial spline give the best approximation to the solution of two-point boundary value problems compared to the approximation shown by Successive Over Relaxation (SOR) method and Gauss-Seidel (GS) method.
Novelty – the performance of CG iterative method is found to be superior in respect of iterations number, execution time and maximum absolute error on various grid sizes.

Type of Paper: Empirical

Keywords: Cubic Non-Polynomial Solution; Two-Point Boundary Value Problems; Conjugate Gradient; Successive Over Relaxation; Gauss-Seidel.

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1. Introduction

There are a vast number of solutions and approaches have been implemented to obtain the approximate solution for solving two-point boundary value problems irrespective of linear point and non-linear point (Kelley, 1995). owing to its broad function in the field of sciences, economics and engineering. Prior to the abundancy of past literatures, today's researchers are triggered to dig more into computation of the early solutions which have led them to initiate various approaches based on modification or enhancement to the previous one. Among several

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of the remarkable methods have been initiated todays are EADM method (Jang, 2008) and PTI method (Chen *et al*., 2006). In 2001, Ha came up with a solution called as nonlinear shooting method followed by another implementation of solution in the latter years which is known as mean weight by Mohsen and El-Gamel (2008). Moreover, Fang *et al.* (2002) emphasized finite difference, finite element and finite volume. Furthermore, another approach which has significant contribution to solve these problems is spline approach as highlighted by Albasiny and Hoskin (1969). In fact, Ramadan *et al.* (2007) also shared mutual view. In response to the evolving approaches from past researches, the present paper aims to solve the system of linear equations generated from discretization of the two-point boundary value problems with cubic non-polynomial spline scheme by using CG, SOR and GS methods. Important to take note that SOR and GS are conducted for comparison purpose. In order to derive the approximation equation, the cubic non-polynomial spline general function is used to discretize these problems based on spline approximation equations. As a result, this spline approximation equation leads to the formation of its corresponding large and sparse linear system.
There are various iterative methods can be used to solve linear system and have been presented and discussed thoroughly by Young (1972, 2014), Hackbusch (2012), Saad (1996), Hestenes and Stiefiel (1952), Kelly (1995) and Burgerscentrum (2011). Previous studies on iterative methods have shown the existence of several families of iterative methods with different concept called as concept of block iteration which introduced by Evans (1985) and have been further explored by Yousif and Evans (1995) and Ibrahim and Abdullah (1995). In regards with the advantages of CG iterative method on solving system of linear equations as proposed by Hestenes and Stiefiel (1952), the present paper aims to conduct this iterative method together with the cubic non-polynomial spline scheme for solving the two-point boundary value problems. As for comparison purpose in term of performance, SOR and GS methods are set as control methods.

2. Methodology

2.1 Two-point boundary value problems

Generally, the equation of two-point boundary value problems can be expressed as follows:

$$y''+f(x)y'+q(x)y=g(x),\ x\in[a,b] \tag{1}$$

subject to the boundary conditions

$$y(a)=A_1,\quad y(b)=A_2 \tag{2}$$

where A_i, $i=1,2$ is constant and functions $f(x)$ and $g(x)$ are known function with boundary $[a,b]$.The functions $f(x)$ and $g(x)$ cannot be created randomly to solve for (1) as the analytical solution for problem (1) is depending on the boundary conditions (2).
Moreover, for the ease of discretizing problem (1) later, the domain of the solution is being restrained to the restriction as shown in Fig. 1, through uniform separation of the set of nodes.

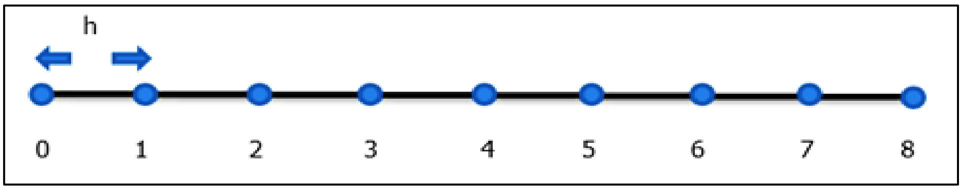

Figure 1. Distribution of node point for domain solution $m=8$

By considering any positive integer $m=2^p$, $p\geq 2$ and letting the solution domain, $[a,b]$ to be divided uniformly into m subinterval, then the length of the uniform subintervals, Δx can be defined as

$$\Delta x = \frac{b-a}{m} = h, \quad n = m-1 \tag{3}$$

As shown by Fig. 1, the grid network has m numbers of interior grid points. A uniform grid network of the solution domain is then developed as in Fig. 2 by using the grid size in Fig 1. Hence, the grid points in the solution domain $[a,b]$ can be labelled as $x_i = a + ih, \quad i = 0, 1, 2, \cdots, m$. Other than that, the values of the function $y(x)$ at the grid points are denoted as $y(x_i) = y_i$. The interior grid points as shown in Fig. 2 are used to formulate and implement GS, SOR and CG iterative methods until the convergence test can be satisfied. After that, the formulation and implementation processes for the three proposed point iterative methods on the same type of deepest node points as well as the convergence test for the iteration, can be done according to the finite grid network as shown in Fig. 1 once the solution of domain for problem (1) has been divided into uniform interval.

2.2 Cubic non-polynomial spline approximation equation

The cubic non-polynomial spline scheme is used to discretize problem (1) in order to construct the cubic non-polynomial approximation equation. The discretization is done by letting $y(x)$ to be the exact solution for problem (1) and S_i be a cubic non-polynomial spline approximation to $y_i = y(x_i)$, which is obtained from the segments of $Q_i(x)$ that passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). Therefore, the non-polynomial spline approximation can be expressed in general form as

$$S(x) = Q_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \cdots, n \tag{4}$$

Then, the general form for the cubic non-polynomial spline from equation (4) is defined in $Q_i(x)$ as

$$Q_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i(x - x_i) + d_i \tag{5}$$

for $i = 0, 1, 2, \cdots, n$ where a_i, b_i, c_i and d_i are constant, and k is the frequency for the trigonometric function. Equation (5) is known as the general form of cubic non-polynomial spline and it interpolates $y(x)$ at the points x_i by depending on k and reducing to cubic spline in $[a,b]$ as $k \to 0$.

As mentioned earlier, the discretization process as shown in Fig. 2 is very important to be conducted first in order to formulate the approximation equation for the cubic non-polynomial spline. In this paper, the approximation of problem (1) has been discretized by using the cubic non-polynomial spline scheme as defined in equation (5).

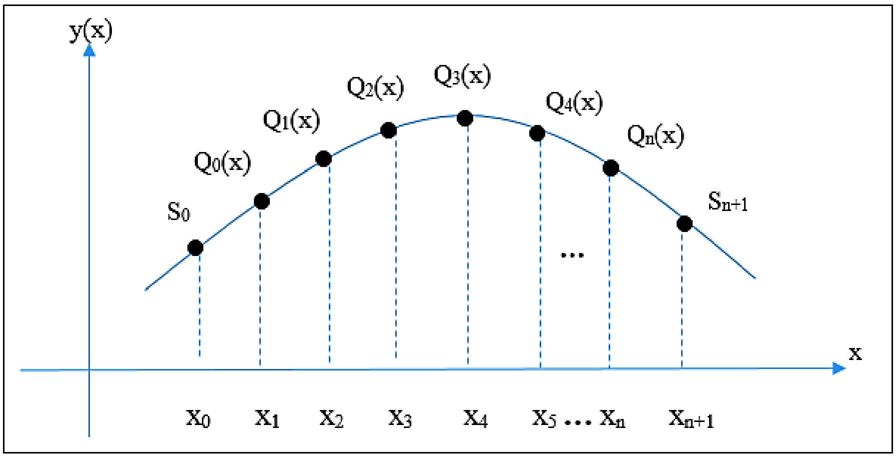

Figure 2. Illustration of cubic non-polynomial spline function for domain solution $m = 8$

Next, y_i is assumed as an accurate solution which obtained from different segments of the spline functions that passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}) . Then, each of the spline functions obtained are defined as follows:

$$Q_i(x_i) = y_{i,} \quad Q_i(x_{i+1}) = y_{i+1,} \quad Q_i'(x_i) = D_{i,} \quad Q_i'(x_{i+1}) = D_{i+1,} \quad Q_i''(x_i) = S_{i,} \quad Q_i''(x_{i+1}) = S_{i+1,}$$

in order to get the expression of constant variables, a_i, b_i, c_i and d_i in form of $y_{i,} y_{i+1,}$ $D_{i,} D_{i+1,} S_{i,}$, S_{i+1} .
After straightforward calculation is done, the value for all the constants a_i, b_i, c_i and d_i are obtained in the following form

$$a_i = h^2 \frac{-S_{i+1} + S_i \cos(\theta)}{\theta^2 \sin(\theta)},$$
$$b_i = -h^2 \frac{S_i}{\theta^2},$$
$$c_i = \frac{y_{i+1} - y_i}{h} + h \frac{(S_{i+1} + S_i)}{\theta^2},$$
$$d_i = y_i + h^2 \frac{S_i}{\theta^2},$$

where $\theta = kh$ and $i = 0,1,2,..,N$.

The condition $Q_{i-1}^m(x) = Q_i^m(x)$ where $m = 0,1$ has been considered after all the points a_i, b_i, c_i and d_i which passing through the point $(x_i y_i)$ are obtained. Then, this part is solved simultaneously to get the following cubic non-polynomial spline approximation equation

$$y_{i-1} - h^2 S_{i-1}(\frac{1}{\theta \sin\theta} - \frac{1}{\theta^2}) - 2y_i - 2h^2 S_i(\frac{1}{\theta^2} - \frac{\cos\theta}{\theta\sin\theta}) + y_{i+1} - 2h^2 S_i(\frac{1}{\theta\sin\theta} - \frac{1}{\theta^2}) = 0 \tag{6}$$

where $i = 1,..., n.$ Equation (6) can be simplified as

$$-y_{i-1} + 2y_i - y_{i+1} + h^2[\alpha S_{i-1} + \beta S_i + \alpha S_{i+1}] = 0 \tag{7}$$

where $\alpha = [\frac{1}{\theta\sin\theta} - \frac{1}{\theta^2}],\ \beta = [\frac{1}{\theta^2} - \frac{\cos\theta}{\theta\sin\theta}]$ and $i = 1,.., N.$

The equation of central finite difference, backward finite difference and forward finite difference schemes can be expressed as

$$S_{i-1} = -f_{i-1}y'_i - q_{i-1}y_{i-1} + g_{i-1,} \quad S_i = -f_i y'_i - q_i y_i + g_{i,} \quad S_{i+1} = -f_{i+1}y'_{i+1} - q_{i+1}y_{i+1} + g_{i+1,} \tag{8}$$

where $y'_i = \frac{y_{i+1} - y_{i-1}}{2h},\ y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h},\ y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}$

and are used to solve (7) by substituting (8) into (7). Then this yields to the equation in the following form

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = F_i, \quad i = 1,2,..., n \tag{9}$$

where

$$a_i = -\mu_0 + \gamma\left(\frac{3p_{i-1}}{2h} - q_{i-1}\right) + \theta\frac{p_i}{2h} - \gamma\frac{p_{i+1}}{2h},\; b_i = 2\mu_0 - \gamma\frac{2p_{i-1}}{h} - \theta q_i + \gamma\frac{2p_{i+1}}{h}$$

$$c_i = -\mu_0 + \gamma\frac{p_{i+1}}{2h} - \theta\frac{p_i}{2h} + \gamma\left(\frac{3p_{i+1}}{2h} - q_{i+1}\right),\; F_i = -\gamma f_{i-1} - \theta f_i \frac{}{2h} + \gamma f_{i+1}$$

$$\gamma = \mu_1 h^2,\quad \phi = 2\mu_2 h^2,\quad \mu_1 = \theta - \sin\theta,\quad \mu_2 = \sin\theta - \theta\cos\theta.$$

By using equation (9), a system of linear equations in a matrix form can be constructed as

$$A\underline{y} = \underline{F} \tag{10}$$

where

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & bn_1 \end{bmatrix}_{(n\times n)},$$

$$\underline{y} = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n \end{bmatrix}^T,$$

$$\underline{F} = \begin{bmatrix} F_1 - a_1 y_0 & F_2 & F_3 & \cdots & F_{n-1} & F_n - c_n y_{n+1} \end{bmatrix}^T,$$

2.3 Derivation of CG method

As mentioned earlier, CG method is an iterative method introduced by Hestenes and Stiefiel (1952) which terminates in at most n steps if there are no rounding-off errors occurred, given that the matrix A in equation (10) is assumed to be symmetric and positive definite. In addition to that, Hestenes and Stiefiel also stated that this method is superior to elimination method such as Gauss elimination method, much simpler to code and less storage space required, the original data can be used to its maximum as the matrix given remain unaltered during the method implementation and an improvement of the solution occur at each step of estimation.
By referring to equation (10) where matrix A is assumed to be n x n symmetrical matrix, then the formulation of CG iterative method can be computed by finding the sequence of n vectors $p_0, p_1, ..., p_{n-1}$ which are the basis in R^n and fulfill the condition as follows:

$$p_i^T A p_j = 0,\quad i \neq j. \tag{11}$$

As for the formulation for method SOR and GS, the matrix A in equation (10) has been decomposed as

$$A = D + L + U \tag{12}$$

where L, D and U are lower triangular matrix, diagonal matrix and upper triangular matrix respectively. Then, the decomposition in equation (12) has been imposed into equation (10). Therefore, the formulation obtained for SOR iterative method is as follows:

$$\underset{\sim}{y}^{(k+1)} = (1-\omega)\,\underset{\sim}{y}^{(k)} + \omega(D+L)^{-1}\left(-U\,\underset{\sim}{y}^{(k)} + \underset{\sim}{F}\right) \tag{13}$$

whereas, the formulation for GS iterative method can be written as

$$\underset{\sim}{y}^{(k+1)} = -(D+L)^{-1}U\,\underset{\sim}{y} + (L+D)^{-1}F \tag{14}$$

In order to facilitate the convergence rate of SOR iterative method, the value for parameter ω must be determined correctly and in practice the range of optimal value for ω is $1 \le \omega < 2$ which obtained by conducting several computer programs. Then, the best approximate value for ω is chosen based on its smallest number of iterations. As for GS iterative method, it can be formed by reducing the SOR iterative method with the value of. $\omega = 1$. Since SOR and GS iterative method are assigned as control methods in this study, thus only the algorithm for CG iterative method is presented, Algorithm 1.

Algorithm 1: CG Scheme

i. Initialize x_0.
ii. Compute the residual $r_0 = f - Ax_0$ and choose a direction of $p_0 = r_0 - fAx_0$.
iii. Obtain the new x_i, $r_i = f - Ax_i$ and the direction p_i then compute the new estimate x_{i+1} and its residual r_{i+1} by using the formulas

$$\alpha_i = \frac{r_i^T r_i}{p_i^T A p_i},$$
$$x_{i+1} = x_i + \alpha_i p_i,$$
$$r_{i+1} = r_i - \alpha_i A p_i.$$

iv. Next find the direction of p_{i+1} by using the formulas and repeat step (iii)

$$p_{i+1} = r_{i+1} - \beta_i p_i \quad \text{where} \quad \beta_i = \frac{r_{i+1}^T r_{i+1}}{r_i^T r_i}$$

v. Check the convergence. If yes, go to step (vi). Otherwise go back to step (iii).
vi. Display approximate solutions.

2.4 Numerical performance analysis

The performance analysis of the cubic non-polynomial spline approximation by using the three proposed point iterative methods are investigated in respect of their iteration numbers, execution time and maximum absolute error. The effectiveness of the three methods is verified by conducting numerical test on the following equation

$$y'' - 4y = 4\cosh(1), \quad x \in [0,1] \tag{15}$$

where the exact solution for problem (15) is given as

$$y(x) = \cosh(2x-1) - \cosh(1).$$

Then, the results for the performance analysis have been tabulated in Table 1.

Table 1. Comparison of iterations number, execution time and maximum absolute error for GS, SOR and CG iterative method

Number of Iterations					
m	**128**	**256**	**512**	**1024**	**2048**
GS	18173	66139	238353	848604	2975185
SOR	382	723	1438	4097	5367
CG	65	129	257	513	1025
Execution Time (Second)					
GS	16.4300	47.1800	169.3099	881.0900	3747.6400
SOR	0.8700	1.6500	2.400	5.6800	7.7200
CG	0.0099	0.0299	0.0700	0.1200	0.2300

Maximum Absolute Error					
GS	9.5665e-06	1.9487e-06	1.2847e-06	7.4088e-06	3.0203e-05
SOR	9.6849e-06	2.4225e-06	6.0855e-07	1.5644e-07	2.6459e-08
CG	9.6845e-06	2.4211e-06	6.0526e-07	1.5132e-07	3.7851e-08

3. Conclusion

The approximate solutions to the two-point boundary value problems have been formulated with non-polynomial spline scheme and solved by using the three proposed methods. Then, summarization withdrawn based on Table 1 enables us to examine which methods give the most favorable approximate solution. Based on the performance analysis in Table 1, SOR iterative method performed better in term of number of iterations, execution time and maximum absolute error compared to the GS iterative method. But then, when comparing GS and SOR iterative methods with the CG iterative method, the performance of CG iterative method is found to be superior in respect of iterations number, execution time and maximum absolute error on various grid sizes.

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